10. Problem sheet for Set Theory, Winter 2012

Prof. Dr. Peter Koepke, Dr. Philipp Schlicht
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Problem 35. Suppose κ is an infinite cardinal. Show that for every $\alpha < \kappa^+$, there is a sequence $(X_n \mid n \in \omega)$ such that $\alpha = \bigcup_{n\in\omega} X_n$ and $otp(X_n) \leq \kappa^n$ (in ordinal exponentiation).

Problem 36. If $(X, <)$ is a linearly ordered set, the *order topology* on X is defined as the topology with basic open sets (a, b), and $[a, b)$ if $a = \min(X)$, $(a, b]$ if $b = \max(X)$, for $a, b \in X$. A topological space X is *compact* if every open cover $(U_\alpha \mid \alpha < \gamma)$ of X (i.e. each U_α is open and $X = \bigcup_{\alpha<\gamma} U_\alpha$) has a finite subcover.

Show that an ordinal α is compact in its order topology if and only if it is a successor or 0.

Problem 37. Suppose κ is an infinite regular cardinal. We say that subsets A, B of κ with $card(A) = card(B) = \kappa$ are *almost disjoint* if $card(A \cap B) < \kappa$. Suppose $(A_\alpha \mid \alpha < \gamma)$ is a sequence of pairwise almost disjoint subsets of κ with $\gamma \leq \kappa$. Show that there is a set $A \subseteq \kappa$ (of cardinality κ) that is almost disjoint from A_α for all $\alpha < \gamma$.

Problem 38. Suppose κ is an infinite cardinal with $2^{<\kappa} = \kappa$. A family $(A_\alpha \mid \alpha < \gamma)$ of subsets of κ with $card(A_\alpha) = \kappa$ for all α is called *almost disjoint* if the sets A_α are pairwise almost disjoint. Show that there is an almost disjoint family of cardinality 2^κ of subsets of κ, by defining a function $f\colon {}^\kappa 2 \to {}^\kappa 2$ such that the values of $f(x)$ code initial segments of x.

There are 6 points for each problem. Please hand in your solutions on Monday, December 17 before the lecture.